

ON C^* -ALGEBRAS RELATED TO CONSTRAINED REPRESENTATIONS OF A FREE GROUP

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ABSTRACT. We consider representations of the free group F_2 on two generators such that the norm of the sum of the generators and their inverses is bounded by $\mu \in [0, 4]$. These μ -constrained representations determine a C^* -algebra A_μ for each $\mu \in [0, 4]$. When $\mu = 4$ this is the full group C^* -algebra of F_2 . We prove that these C^* -algebras form a continuous bundle of C^* -algebras over $[0, 4]$ and calculate their K -groups.

1. INTRODUCTION

The aim of this paper is to study certain family of C^* -algebras related to representations of a free group with a given bound for the norm of the sum of the generating elements.

Let Γ be a discrete group. If we consider different sets of unitary representations of Γ (all representations in this paper are unitary ones), they lead to different group C^* -algebras of Γ . For example, the full group C^* -algebra of Γ , denoted by $C^*(\Gamma)$, is the closure of the group ring $\mathbb{C}[\Gamma]$ with respect to the norm induced by the universal representation (or, equivalently, by all representations); while the reduced group C^* -algebra of Γ , denoted by $C_r^*(\Gamma)$, is the closure of $\mathbb{C}[\Gamma]$ with respect to the norm induced by the regular representation. Here we consider some special classes of representations for the free group on two generators in order to obtain the corresponding C^* -algebras. These classes are related to the special element x of the group ring — the sum of all generators and their inverses, sometimes called an *averaging operator*. This element plays an important role in research related to groups and their C^* -algebras. For example, amenability of Γ is equivalent to $\|\lambda(x)\| = n$, where λ is the regular representation of Γ (we use the same notation for representations of groups and of their group rings and C^* -algebras) and n is the number of summands in x (twice the number of generators). Property (T) for Γ is equivalent to existence of a spectral gap near n in the spectrum of $\pi(x)$ for the universal representation π [1].

Let u and v denote the two generators of the free group F_2 . Then $x = u + u^{-1} + v + v^{-1} \in \mathbb{C}[\Gamma]$.

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Definition 1.1. For $\mu \in [0, 4]$, a representation $\pi : F_2 \rightarrow \mathcal{U}(\mathcal{H}_\pi)$ is called a μ -constrained representation if

$$(1.1) \quad \|\pi(x)\| = \|\pi(u) + \pi(u)^* + \pi(v) + \pi(v)^*\| \leq \mu.$$

Given $0 \leq \mu \leq 4$, the assignment $u, v \mapsto (\mu/4 \pm i\sqrt{1 - (\mu/4)^2})I$, where I is the identity operator on any (in particular, one-dimensional) Hilbert space, gives rise to a μ -constrained representation of F_2 for any $\mu \in [0, 4]$. This shows that μ -constrained representations exist. Actually there are abundant. For example, all representations of F_2 are 4-constrained ones, in which case there is actually no constraint at all. Moreover, if π is a μ -constrained representation and $\mu \leq \mu' \leq 4$, then π is also a μ' -constrained one. Let Π_μ denote the set of all μ -constrained representations, then $\Pi_{\mu_1} \subseteq \Pi_{\mu_2}$ if $0 \leq \mu_1 \leq \mu_2 \leq 4$. The one-dimensional example above shows also that if $\mu_1 \neq \mu_2$ then Π_{μ_1} is strictly smaller than Π_{μ_2} . Note that Π_4 consists of all representations of F_2 .

As in the case of $C^*(F_2)$, we first define a (semi)norm $\|\cdot\|_\mu$ over $\mathbb{C}[F_2]$ induced by Π_μ and then complete $\mathbb{C}[F_2]$ with respect to $\|\cdot\|_\mu$, thus obtaining the corresponding group C^* -algebra A_μ .

Definition 1.2. For $a \in \mathbb{C}[F_2]$, $\mu \in [0, 4]$, set

$$(1.2) \quad \|a\|_\mu := \sup_{\pi \in \Pi_\mu} \|\pi(a)\|.$$

Remark 1.3. Since $\|a\|_\mu \leq \|a\|_4 = \|a\|_{\max}$, where $\|\cdot\|_{\max}$ is the norm on $\mathbb{C}[F_2]$ induced by the universal representation, it is clear that $\|\cdot\|_\mu$ is bounded. Moreover, $\|\cdot\|_{\mu_1} \leq \|\cdot\|_{\mu_2}$ if $0 \leq \mu_1 \leq \mu_2 \leq 4$. As we use only unitary representations, this is a C^* -seminorm.

Set $\mathcal{N}_\mu = \{a \in \mathbb{C}[F_2] : \|a\|_\mu = 0\}$ and complete $\mathbb{C}[F_2]/\mathcal{N}_\mu$ with respect to $\|\cdot\|_\mu$ (which is already a norm there). Let us denote this completion by A_μ . It is obviously a C^* -algebra for any $\mu \in [0, 4]$. Our aim is to study the family of C^* -algebras A_μ .

Remark 1.4. Note that A_μ can be defined as a *universal* C^* -algebra generated by two unitaries, u, v satisfying a single relation $\|u + u^* + v + v^*\| \leq \mu$.

Proposition 1.5. For any $0 \leq \mu_1 \leq \mu_2 \leq 4$, the identity map on $\mathbb{C}[F_2]$ extends to a surjective $*$ -homomorphism $A_{\mu_2} \rightarrow A_{\mu_1}$.

Proof. Since $\mathcal{N}_{\mu_2} \subset \mathcal{N}_{\mu_1}$, the identity map on $\mathbb{C}[F_2]$ gives rise to a map $\mathbb{C}[F_2]/\mathcal{N}_{\mu_2} \rightarrow \mathbb{C}[F_2]/\mathcal{N}_{\mu_1}$, which extends to a $*$ -homomorphism from A_{μ_2} to A_{μ_1} by continuity. Since the range of this $*$ -homomorphism is dense in A_{μ_1} , it is surjective. \square

Note that A_4 is isomorphic to the full group C^* -algebra $C^*(F_2)$. Later on we shall give a description for A_0 . For $2\sqrt{3} \leq \mu \leq 4$ the identity map on $\mathbb{C}[F_2]$ extends to a surjective $*$ -homomorphism from A_μ to the reduced group C^* -algebra $C_r^*(F_2)$ [4].

The aim of this paper is to study the family of C^* -algebras A_μ . In the next section we show that this family is a continuous bundle of C^* -algebras and then we identify A_0 as a certain amalgamated free product. Finally, following Cuntz [2], we calculate the K -theory groups for A_μ and show that they don't depend on μ .

2. CONTINUITY OF A_μ

If μ_1 is close to μ_2 then one would expect that A_{μ_1} and A_{μ_2} are close to each other. In other words, there is some kind of “continuity” of A_μ with respect to μ . In order to characterize such “continuity”, we use the notion of continuous bundle of C^* -algebras due to Dixmier.

Let I be a locally compact Hausdorff space and let $\{A(x)\}_{x \in I}$ be a family of C^* -algebras. Denote by $\prod_{x \in I} A(x)$ the set of functions $a = a(x)$ defined on I and such that $a(x) \in A(x)$ for any $x \in I$.

Definition 2.1 ([3]). Let $\mathcal{A} \subset \prod_{x \in I} A(x)$ be a subset with the following properties:

- (i) \mathcal{A} is a $*$ -subalgebra in $\prod_{x \in I} A(x)$,
- (ii) for any $x \in I$ the set $\{a(x) : a \in \mathcal{A}\}$ is dense in the algebra $A(x)$,
- (iii) for any $a \in \mathcal{A}$ the function $x \mapsto \|a(x)\|$ is continuous,
- (iv) let $a \in \prod_{x \in I} A(x)$, if for any $x \in I$ and for any $\varepsilon > 0$ one can find such $a' \in \mathcal{A}$ such that $\|a(x) - a'(x)\| < \varepsilon$ in some neighborhood of the point x , then one has $a \in \mathcal{A}$.

Then the triple $(A(x), I, \mathcal{A})$ is called a continuous bundle of C^* -algebras.

Let $I = [0, 4]$, $A = C(I, C^*(F_2))$ and let $B = \{f \in A : \|f(\mu)\|_\mu = 0, \forall \mu \in I\}$. It is clear that B is a closed ideal of A , with the quotient map $q : A \rightarrow A/B$. Define the map $\iota : A/B \rightarrow \prod_{\mu \in I} A_\mu$ by $\iota(b)(\mu) = q_\mu(a(\mu))$, where $b \in A/B$ and $a \in A$ such that $b = q(a)$, $q_\mu : C^*(F_2) = A_4 \rightarrow A_\mu$ is the quotient map. It is simple to check that ι is well-defined and injective, so from now on we treat A/B as a subalgebra of $\prod_{\mu \in I} A_\mu$. In order to prove that $(A_\mu, I, A/B)$ is a continuous bundle of C^* -algebras, we need some lemmas.

Lemma 2.2. *For any $a \in C^*(F_2)$, the function $N_a : I \rightarrow \mathbb{R}_+$ defined by $\mu \mapsto \|a\|_\mu$ is continuous.*

Proof. Given any fixed $\mu_0 \in I$, we will prove that N_a is continuous at μ_0 in two steps: N_a is left and right continuous at μ_0 , respectively.

Step 1. Note that N_a is a non-decreasing function, so $l = \lim_{\mu \rightarrow \mu_0^-} N_a(\mu)$ exists. Assume that $l < N_a(\mu_0)$, then there must exist a representation π of F_2 such that $\|\pi(u + u^* + v + v^*)\| = \mu_0$ and $l < \|\pi(a)\| \leq N_a$.

Let us first give a family of Borel functions $\{f_t : S^1 \rightarrow S^1\}_{t \in [0,1]}$ as follows:

$$f_t(e^{i\theta}) = \begin{cases} e^{i \arccos((1-t) \cos \theta)}, & \theta \in [0, \pi] \\ e^{-i \arccos((1-t) \cos \theta)}, & \theta \in (-\pi, 0) \end{cases}$$

Applying Borel functional calculus of f_t to $\pi(u)$ and $\pi(v)$, we get a new representation π_t of F_2 which is defined by $u \mapsto f_t(\pi(u))$ and $v \mapsto f_t(\pi(v))$, and $\{\pi_t\}_{t \in [0,1]}$ is a continuous family of representations. Since $f_t(z) + f_t(\bar{z}) = (1-t)(z + \bar{z})$, we have $\pi_t(u + u^* + v + v^*) = f_t(\pi(u)) + f_t(\pi(u^*)) + f_t(\pi(v)) + f_t(\pi(v^*)) = (1-t)(u + u^* + v + v^*)$. Hence $\|\pi_t(u + u^* + v + v^*)\| < \mu_0$. Meanwhile, $\|\pi_t(a)\|$ varies also continuously, which contradicts the assumption.

Step 2. Assume that, for some $a \in \mathbb{C}[F_2]$, N_a is not continuous at μ_0 from the right, i.e., $N_a(\mu_0) < \lim_{\mu \rightarrow \mu_0^+} N_a(\mu) = r$. Then there must exist a family of representations $\{\pi_n : F_2 \rightarrow \mathcal{U}(\mathcal{H}_n)\}_{n \in \mathbb{N}}$ such that $\{\|\pi_n(u + u^* + v + v^*)\|\}_{n \in \mathbb{N}}$ is a decreasing sequence convergent to μ_0 and $\lim_{n \rightarrow \infty} \|\pi_n(a)\| = r$.

Let $\prod_{n \in \mathbb{N}} B(\mathcal{H}_n)$ be the C^* -algebra of all sequences $b = (b_1, b_2, \dots)$, $b_n \in B(\mathcal{H}_n)$, such that $\|b\| := \sup_{n \in \mathbb{N}} \|b_n\| < \infty$. Let $\oplus_{n \in \mathbb{N}} B(\mathcal{H}_n)$ be the ideal of $\prod_{n \in \mathbb{N}} B(\mathcal{H}_n)$ that consists of sequences (b_1, b_2, \dots) such that $\lim_{n \rightarrow \infty} \|b_n\| = 0$. Then $\prod_{n \in \mathbb{N}} B(\mathcal{H}_n) / \oplus_{n \in \mathbb{N}} B(\mathcal{H}_n)$ is a quotient C^* -algebra. By Gelfand-Naimark-Segal theorem, there exists a faithful representation $\rho : \prod_{n \in \mathbb{N}} B(\mathcal{H}_n) / \oplus_{n \in \mathbb{N}} B(\mathcal{H}_n) \rightarrow B(\mathcal{H})$ for some Hilbert space \mathcal{H} . Let $q : \prod_{n \in \mathbb{N}} B(\mathcal{H}_n) \rightarrow \prod_{n \in \mathbb{N}} B(\mathcal{H}_n) / \oplus_{n \in \mathbb{N}} B(\mathcal{H}_n)$ be the canonical quotient map. Note that, if $b = (b_1, b_2, \dots) \in \prod_{n \in \mathbb{N}} B(\mathcal{H}_n)$, $\|(\rho \circ q)(b)\| = \|q(b)\| = \limsup_{n \rightarrow \infty} \|b_n\|$.

Let π_∞ be the representation of F_2 defined by $u \mapsto (\rho \circ q)((\pi_1(u), \pi_2(u), \dots))$ and $v \mapsto (\rho \circ q)((\pi_1(v), \pi_2(v), \dots))$. Then $\|\pi_\infty(u + u^* + v + v^*)\| = \limsup_{n \rightarrow \infty} \|\pi_n(u + u^* + v + v^*)\| = \mu_0$, so π_∞ is a μ_0 -constrained representation of F_2 . But $\|\pi_\infty(a)\| = \limsup_{n \rightarrow \infty} \|\pi_n(a)\| = r > \|a\|_{\mu_0}$, which is a contradiction. \square

Recall that $B = \{f \in C(I, C^*(F_2)) : \|f(\mu)\|_\mu = 0 \text{ for any } \mu \in I\}$.

Lemma 2.3. *Set $I_\mu = \{a \in C^*(F_2) : \|a\|_\mu = 0\}$. Then $\{g(\mu_0) : g \in B\} = I_{\mu_0}$ for any $\mu_0 \in I$.*

Proof. From the definition of B , it is obvious that $\{f(\mu_0) : f \in B\} \subseteq I_{\mu_0}$, thus we just need to prove the converse inclusion. An easy observation implies that it suffices to prove this inclusion for positive elements of I_{μ_0} .

Let $a \in I_{\mu_0}$ be positive. We have to find $g \in B$ such that $g(\mu_0) = a$. Define a family of continuous functions by

$$f_\mu(t) = \begin{cases} 0, & t \in (-\infty, \|a\|_\mu] \\ t - \|a\|_\mu, & t \in (\|a\|_\mu, \infty). \end{cases}$$

As $\|a\|_\mu = 0$ for $\mu \leq \mu_0$, so $f_\mu(a) = a$ for $\mu \leq \mu_0$. It follows from Lemma 2.2 that f_μ is continuous in μ . Define a function $g : I \rightarrow C^*(F_2)$ by $g(\mu) = f_\mu(a)$. Then $g \in A = C(I, C^*(F_2))$ and $g(\mu_0) = a \in I_{\mu_0}$.

Let $q_\mu : C^*(F_2) \rightarrow C^*(F_2)/I_\mu \cong A_\mu$ denote the quotient map. As $\|q_\mu(a)\| = \|a\|_\mu$, so $q_\mu(f_\mu(a)) = f_\mu(q_\mu(a)) = 0$, thus $g(\mu) = f_\mu(a) \in I_\mu$, hence $g \in B$. \square

Since we have treated A/B as a subalgebra of $\prod_{\mu \in I} A_\mu$, for any $b \in A/B$, besides the quotient norm, we can also treat b as a function defined on I and take the supremum norm. The following lemma asserts that these two norms coincide.

Lemma 2.4. *Let $a \in A$, $b = q(a) \in A/B$. Set*

$$\|b\|_1 = \inf_{g \in B} \|a + g\| = \inf_{g \in B} \sup_{\mu \in I} \|a(\mu) + g(\mu)\|$$

and

$$\|b\|_2 = \sup_{\mu \in I} \inf_{g \in B} \|a(\mu) + g(\mu)\| = \sup_{\mu \in I} \|a(\mu)\|_\mu \text{ (by Lemma 2.3).}$$

Then $\|b\|_1 = \|b\|_2$.

Proof. This follows from uniqueness of a C^* -norm on the C^* -algebra A/B . \square

Theorem 2.5. *$(A_\mu, I, A/B)$ is a continuous bundle of C^* -algebras.*

Proof. Let us check the conditions from the definition of a continuous bundle of C^* -algebras one by one.

(i) and (ii) are obviously satisfied and $\{a(\mu) : a \in A/B\}$ equals A_μ .

For any $b \in A/B$ with $b = q(a)$ where $a \in A$, given $\mu, \mu' \in I$,

$$\begin{aligned} & | \|b(\mu')\|_{\mu'} - \|b(\mu)\|_{\mu} | \\ & \leq | \|a(\mu')\|_{\mu'} - \|a(\mu)\|_{\mu} | \\ & \leq | \|a(\mu')\|_{\mu'} - \|a(\mu)\|_{\mu'} | + | \|a(\mu)\|_{\mu'} - \|a(\mu)\|_{\mu} | \\ & \leq \|a(\mu') - a(\mu)\|_{\mu'} + | \|a(\mu)\|_{\mu'} - \|a(\mu)\|_{\mu} | \\ & \leq \|a(\mu') - a(\mu)\|_{\max} + | \|a(\mu)\|_{\mu'} - \|a(\mu)\|_{\mu} |, \end{aligned}$$

If μ' is close to μ then $\|a(\mu') - a(\mu)\|_{\max}$ is small because the function $\mu \mapsto a(\mu)$ is continuous, and $\|a(\mu)\|_{\mu'} - \|a(\mu)\|_{\mu}$ is small due to Lemma 2.2, therefore, the map $\mu \mapsto \|b(\mu)\|_{\mu}$ is continuous, i.e., (iii) is satisfied.

Suppose $z \in \prod_{\mu \in I} A_\mu$ such that for every $\mu \in I$ and every $\varepsilon > 0$, there exists an $b \in A/B$ such that $\|z(\mu) - b(\mu)\| \leq \varepsilon$ in some neighborhood U_μ of μ . Thus we obtain an open covering $\{U_\mu\}_{\mu \in I}$ of I . Let $\{U_i\}_{i=1}^p$ be its finite sub-covering and let (η_1, \dots, η_p) be a continuous partition of unity in I subordinate to the covering $\{U_i\}_{i=1}^p$. Then

$$\|z(\mu) - \eta_1(\mu)b_1(\mu) - \dots - \eta_p(\mu)b_p(\mu)\| \leq \varepsilon, \text{ for any } \mu \in I,$$

or equivalently,

$$\|z - \eta_1 b_1 - \dots - \eta_p b_p\| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\eta_i b_i$ belongs to A/B and A/B is norm closed, we have $z \in A/B$. So (iv) is satisfied. \square

3. A_0 AS AN AMALGAMATED FREE PRODUCT

Here we identify A_0 as an amalgamated product of C^* -algebras.

Recall that, given C^* -algebras A_1, A_2 and B and embeddings $i_k : B \rightarrow A_k$, $k = 1, 2$, the *amalgamated free product* is the C^* -algebra, denoted $A_1 *_B A_2$, together with embeddings $j_k : A_k \rightarrow A_1 *_B A_2$, satisfying $j_1 \circ i_1 = j_2 \circ i_2$, such that the following holds: if $\phi_k : A_k \rightarrow A$, $k = 1, 2$, are $*$ -homomorphisms with $\phi_1 \circ i_1 = \phi_2 \circ i_2$ then there is a unique $*$ -homomorphism $\phi : A_1 *_B A_2 \rightarrow A$ such that $\phi \circ j_k = \phi_k$, $k = 1, 2$. The $*$ -homomorphism ϕ induced by ϕ_1 and ϕ_2 will sometimes be denoted by $\phi_1 *_B \phi_2$.

Let $p : S^1 \rightarrow [-1, 1]$ be the projection of the circle $x^2 + y^2 = 1$ onto the x axis. It induces an inclusion $i_1 : C[-1, 1] \rightarrow C(S^1)$ such that $i_1(\text{id}) = z + \bar{z}$, where $z = x + iy$ is the coordinate on S^1 and id is the identity function on $C[-1, 1]$. Let $\tau : C[-1, 1] \rightarrow C[-1, 1]$ be the flip automorphism, which changes the orientation of the interval and is given by $\text{id} \mapsto -\text{id}$. Set $i_2 = i_1 \circ \tau$. Then $i_2(\text{id}) = -(z + \bar{z})$.

The inclusions i_1 and i_2 of $C[-1, 1]$ into $C(S^1)$ give us the amalgamated free product $D = C(S^1) *_C[-1, 1] C(S^1)$.

Lemma 3.1. *C^* -algebras A_0 and D are isomorphic.*

Proof. Recall that A_0 is a universal C^* -algebra generated by two unitaries, u and v , with a single relation $u + u^* = -(v + v^*)$.

Let \tilde{u}, \tilde{v} be generators for the two copies of $C(S^1)$. Define $\varphi_k : C(S^1) \rightarrow A_0$, $k = 1, 2$, by $\varphi_1(\tilde{u}) = u$, $\varphi_2(\tilde{v}) = v$. Then $\varphi_1 \circ i_1 = \varphi_2 \circ i_2$, hence the maps φ_k give rise to a $*$ -homomorphism $D \rightarrow A_0$.

Using universality of A_4 , we can construct a $*$ -homomorphism $\psi : A_4 \rightarrow D$ by setting $\psi(u) = \tilde{u} * 1$, $\psi(v) = 1 * \tilde{v}$. Note that A_0 is the quotient of A_4 under a single relation $u + u^* = -(v + v^*)$, and $\psi(u + u^*) = -\psi(v + v^*)$, therefore, ψ factorizes through A_0 , thus giving a $*$ -homomorphism from A_0 to D .

The two $*$ -homomorphisms $D \rightarrow A_0$ and $A_0 \rightarrow D$ are obviously inverse to each other, hence the two C^* -algebras are isomorphic. \square

Now we may apply the K -theory exact sequence for amalgamated free products due to Cuntz [2]:

$$\begin{array}{ccccc} K_0(C[-1, 1]) & \xrightarrow{(i_1, i_2)} & K_0(C(S^1)) \oplus K_0(C(S^1)) & \xrightarrow{j_1 - j_2} & K_0(A_0) \\ \uparrow & & & & \downarrow \\ K_1(A_0) & \xleftarrow{j_1 - j_2} & K_1(C(S^1)) \oplus K_1(C(S^1)) & \xleftarrow{(i_1, i_2)} & K_1(C[-1, 1]) \end{array}$$

Corollary 3.2. (i) $K_0(A_0) \cong \mathbb{Z}$ and is generated by the class [1] of unit element;

(ii) $K_1(A_0) \cong \mathbb{Z}^2$ and is generated by $[u]$ and $[v]$, which are considered as elements of the first and the second copy of $C(S^1)$ respectively.

4. K -GROUPS OF A_μ

In [2], J. Cuntz proved that $K_0(C^*(F_2)) \cong \mathbb{Z}$, $K_1(C^*(F_2)) \cong \mathbb{Z}^2$. Here we use his method to calculate the K -groups for A_μ , $0 \leq \mu < 4$.

Remark 4.1. From Corollary 3.2 we can get some information about K -groups of A_μ ($0 < \mu < 4$). Since the quotient map $A_4 \rightarrow A_0$ factorizes through A_μ and induces an isomorphism in K -theory, we may conclude that $K_*(A_\mu)$ contains $K_*(A_4)$ as a direct summand.

In Section 1 we show that A_μ possesses certain continuity with respect to μ , together with the fact that the K -groups of A_0 and $C^*(F_2)$ are the same, it would be reasonable to conjecture that all A_μ ($0 \leq \mu \leq 4$) have the *same* K -groups. Below we give a proof of this conjecture. The idea of the proof is taken from [2] (cf. Appendix in [5]): to construct a homotopy between the universal representation of F_2 and the trivial representation. But the trivial representation is not constrained for any $\mu < 4$, so we have to replace it by some other representation.

Theorem 4.2. *The quotient map $A_4 \rightarrow A_\mu$ induces an isomorphism of their K_* -groups.*

Proof. Let $B = C(S^1 \vee S^1)$ be the C^* -algebra of continuous functions on the wedge $S^1 \vee S^1$ of two circles. This is the algebra of pairs of functions (f, g) , $f, g \in C(S^1)$ such that $f(1) = g(1)$, where $1 \in S^1$ is the common point of the two circles (we consider the circle as the subset of the complex plane given by $|z| = 1$). Then $K_0(B) \cong \mathbb{Z}$, $K_1(B) \cong \mathbb{Z}^2$.

Set $\alpha(z) = -\operatorname{Re} z + i|\operatorname{Im} z|$. Since $|\alpha(z)| = 1$, this is a function from S^1 to itself with the trivial winding number (equivalently, the trivial homotopy class). Note that $\operatorname{Re}(z + \alpha(z)) = 0$.

For each μ we define $*$ -homomorphisms $\phi : A_\mu \rightarrow M_2(B)$ and $\psi : B \rightarrow M_2(A_\mu)$ as follows,

$$\text{Set } \psi : (z, 1) \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}; (1, z) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}.$$

Note that ψ defines a $*$ -homomorphism from B to $M_2(A_\mu)$ for $\mu = 4$, hence one can pass to the quotient to obtain a $*$ -homomorphism to $M_2(A_\mu)$ for arbitrary μ .

$$\text{Set } \phi : u \mapsto \begin{pmatrix} (z, 1) & (0, 0) \\ (0, 0) & (-1, \alpha(z)) \end{pmatrix}; v \mapsto \begin{pmatrix} (\alpha(z), -1) & (0, 0) \\ (0, 0) & (1, z) \end{pmatrix}.$$

Note that $\phi(u + u^* + v + v^*) = \begin{pmatrix} (0, 0) & (0, 0) \\ (0, 0) & (0, 0) \end{pmatrix}$, so ϕ is well-defined as a $*$ -homomorphism from A_0 to $M_2(B)$. Then it is well-defined for any μ .

For the composition $\psi \circ \phi : A_\mu \rightarrow M_4(A_\mu)$, one has

$$(\psi \circ \phi)(u) = \begin{pmatrix} u & & & \\ & 1 & & \\ & & -1 & \\ & & & \alpha(v) \end{pmatrix}; \quad (\psi \circ \phi)(v) = \begin{pmatrix} \alpha(u) & & & \\ & -1 & & \\ & & 1 & \\ & & & v \end{pmatrix}.$$

Set

$$V_t = \begin{pmatrix} \cos t & 0 & 0 & \sin t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin t & 0 & 0 & \cos t \end{pmatrix} \begin{pmatrix} \alpha(u) & & & \\ & -1 & & \\ & & 1 & \\ & & & v \end{pmatrix} \begin{pmatrix} \cos t & 0 & 0 & -\sin t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin t & 0 & 0 & \cos t \end{pmatrix},$$

$t \in [0, \pi/2]$. Then one can define a homotopy of $*$ -homomorphisms $\lambda_t : A_\mu \rightarrow M_4(A_\mu)$ by

$$\lambda_t(u) = \psi \circ \phi(u); \quad \lambda_t(v) = V_t.$$

Indeed, direct calculation shows that

$$\begin{aligned} \|\lambda_t(x)\| &= \left\| \begin{pmatrix} \sin^2 t \cdot x & 0 & 0 & \sin t \cos t \cdot x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sin t \cos t \cdot x & 0 & 0 & -\sin^2 t \cdot x \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} \sin^2 t & \sin t \cos t \\ \sin t \cos t & -\sin^2 t \end{pmatrix} \right\| \cdot \|x\| = \sin t \cdot \|x\| \leq \|x\| \leq \mu, \end{aligned}$$

where $x = u + u^* + v + v^*$, hence λ_t is continuous for any $t \in [0, \pi/2]$.

Then λ_0 and $\lambda_{\pi/2}$ induce the same map for the K -theory. At the end-points one has $\lambda_0 = \psi \circ \phi$ and $\lambda_{\pi/2} = \operatorname{id}_{A_\mu} \oplus \tau_1 \oplus \tau_2 \oplus \tau_3$, where $\tau_1(u) = 1_{A_\mu}$, $\tau_1(v) = -1_{A_\mu}$; $\tau_2(u) = -1_{A_\mu}$, $\tau_2(v) = 1_{A_\mu}$; $\tau_3(u) = \alpha(v)$, $\tau_3(v) = \alpha(u)$.

Let $\tau : A_\mu \rightarrow A_\mu$ be a $*$ -homomorphism given by $\tau(u) = \tau(v) = i \cdot 1_{A_\mu}$. The formulas $u_t = -t \operatorname{Re} v + i\sqrt{1 - t^2(\operatorname{Re} v)^2}$, $v_t = -t \operatorname{Re} u + i\sqrt{1 - t^2(\operatorname{Re} u)^2}$, $t \in [0, 1]$, provide a homotopy connecting τ_3 and τ . Similarly, τ_1 and τ_2 are homotopic to τ due to the homotopies $u_t = (\pm \cos t + i \sin t) \cdot 1_{A_\mu}$, $v_t = (\mp \cos t + i \sin t) \cdot 1_{A_\mu}$, $t \in [0, \pi/2]$. All these homotopies satisfy the constraint $\|u_t + u_t^* + v_t + v_t^*\| \leq \mu$ when $\|u + u^* + v + v^*\| \leq \mu$.

Thus, for the induced maps in K_* -groups one has $(\psi \circ \phi)_* = \text{id}_{K_*(A_\mu)} + 3\tau_*$, or, equivalently,

$$\text{id}_{K_*(A_\mu)} = (\psi \circ \phi)_* - 3\tau_*.$$

Note that τ factorizes through \mathbb{C} : $\tau : A_\mu \rightarrow \mathbb{C} \rightarrow A_\mu$. Therefore, for K_1 , the map $\tau_* : K_1(A_\mu) \rightarrow K_1(A_\mu)$ is zero (as $K_1(\mathbb{C}) = 0$), so $\text{id}_{K_1(A_\mu)} = (\psi \circ \phi)_*$.

For any $\mu \in (0, 4)$, consider the commuting diagram

$$\begin{array}{ccccc} K_1(A_4) & \xlongequal{\quad} & K_1(A_4) & & \\ \downarrow & \searrow & \nearrow & \downarrow & \\ K_1(A_\mu) & \xrightarrow{\phi_*} & K_1(B) & \xrightarrow{\psi_*} & K_1(A_\mu) \\ \downarrow & \nearrow & \searrow & \downarrow & \\ K_1(A_0) & \xlongequal{\quad} & K_1(A_0) & & \end{array}$$

where the diagonal arrows are isomorphisms and the compositions of the vertical arrows are identity maps. The latter implies that the map $K_1(A_4) \rightarrow K_1(A_\mu)$ is injective. If it is not surjective, there would exist some element in $K_1(A_\mu)$ that doesn't come from $K_1(A_4)$, but this contradicts $\text{id}_{K_1(A_\mu)} = \psi_* \circ \phi_*$. Thus, the map $K_1(A_4) \rightarrow K_1(A_\mu)$ induced by the quotient map is an isomorphism.

As the map $\tau_* : K_0(A_\mu) \rightarrow K_0(A_\mu)$ is not trivial, the case of K_0 is slightly more difficult, and we deal with it below.

Recall that τ factorizes through \mathbb{C} . Let $\rho : A_\mu \rightarrow \mathbb{C}$ denote the character such that $\tau = \iota \circ \rho$, where $\iota : \mathbb{C} \rightarrow A_\mu$ is the canonical inclusion of scalars, $\iota(\lambda) = \lambda \cdot 1_{A_\mu}$. Then $\rho(u) = \rho(v) = i$.

Note that the composition $K_0(\mathbb{C}) \xrightarrow{\iota_*} K_0(A_\mu) \xrightarrow{\rho_*} K_0(\mathbb{C})$ is the identity map on $K_0(\mathbb{C})$. Thus $K_0(A_\mu) = K_0(\mathbb{C}) \oplus \ker \tau_*$.

Let $\sigma : B \rightarrow \mathbb{C}$ be a $*$ -homomorphism defined by $\sigma((z, 1)) = \sigma((1, z)) = i$. Then $\sigma((-1, \alpha(z))) = \alpha(i) = i$. Therefore, $\sigma(\phi(u)) = \sigma(\phi(v)) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, hence $\sigma \circ \phi = \rho \oplus \rho$.

Let $p \in K_0(A_\mu)$, $p \in \ker \rho_* = \ker \tau_*$. Then $(\sigma \circ \phi)_*(p) = 2\rho_*(p) = 0$. As $\sigma_* : K_0(B) \rightarrow K_0(\mathbb{C})$ is an isomorphism, so $p \in \ker \phi_*$.

Since $\text{id}_{K_0(A_\mu)} = (\psi \circ \phi)_* - 3\tau_*$,

$$p = (\psi \circ \phi)_*(p) - 3\tau_*(p) = \psi_*(\phi_*(p)) - 3\iota_*(\rho_*(p)) = 0,$$

hence $\ker \tau_* = 0$, $K_0(A_\mu) \cong \mathbb{Z}$ (generated by $[1_{A_\mu}]$).

□

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